

**\* Unitarity and vector meson contributions  
to  $K^+ \rightarrow \pi^+ \gamma \gamma$**

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**Abstract**

We compute the one-loop unitarity corrections  $\mathcal{O}(p^6)$  from  $K^+ \rightarrow \pi^+ \pi^+ \pi^-$  to  $K^+ \rightarrow \pi^+ \gamma \gamma$  and we find that they are relevant, increasing the leading order prediction for the width in a 30 – 40%. The contributions of local  $\mathcal{O}(p^6)$  amplitudes, generated by vector meson exchange, are discussed in several models and we conclude that the vector resonance contribution should be negligible compared to the unitarity corrections.

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# 1 Introduction

The phenomenology of radiative non-leptonic kaon decays provides crucial tests for the ability of Chiral Perturbation Theory ( $\chi PT$ ) [1] to explain weak low-energy processes and interesting possibilities to study CP violation in these channels.  $\chi PT$  is a natural framework that embodies together an effective theory (satisfying the basic chiral symmetry of QCD) and a perturbative Feynman–Dyson expansion. Its success in the study of radiative non-leptonic kaon decays has been remarkable (see [2, 3] and references therein). Of course there are still open problems, but upcoming experiments should improve our phenomenological knowledge of these decays, in particular on  $K \rightarrow \pi\gamma\gamma$ ,  $K \rightarrow \pi\ell^+\ell^-$ ,  $K_L \rightarrow \ell^+\ell^-$  or  $K_L \rightarrow \gamma\ell^+\ell^-$ .

$K_L \rightarrow \pi^0\gamma\gamma$  is a very interesting channel by itself as a  $\chi PT$  test and also in order to establish the relative role of the CP conserving contribution to  $K_L \rightarrow \pi^0 e^+ e^-$  versus the CP violating contributions [2, 3, 4, 5, 6, 7, 8].  $K_L \rightarrow \pi^0\gamma\gamma$  has no  $\mathcal{O}(p^2)$  tree level contribution since the external particles are neutral. For this same reason there are no  $\mathcal{O}(p^4)$  counterterms and the chiral meson loops are finite at this order [9]. Consequently  $\chi PT$  gives an unambiguous prediction. Furthermore only an helicity suppressed amplitude for  $K_L \rightarrow \pi^0 e^+ e^-$  is generated, which is small compared to the CP violating ones. At higher orders in  $\chi PT$  a new invariant amplitude, generating an unsuppressed helicity amplitude to  $K_L \rightarrow \pi^0 e^+ e^-$ , appears. As we will see in detail later there is a specific kinematical region in the diphoton invariant mass spectra of the  $K \rightarrow \pi\gamma\gamma$  processes where only this last amplitude contributes. This is the region where both photons are nearly collinear, i.e. at small diphoton invariant mass. The experimental results showed [10] that while the  $\mathcal{O}(p^4)$  predicted spectrum looked in good agreement with the experiment, finding no contribution at small diphoton invariant mass, the predicted branching ratio was underestimated by a factor slightly bigger than two, two sigmas away from the experimental result.

This fact prompted to several authors to consider higher chiral order corrections: i) Vector Meson Exchange contributions to the local  $\mathcal{O}(p^6)$  amplitude [7, 8], ii) Unitarity corrections from  $K_L \rightarrow \pi^0\pi^+\pi^-$  [11, 12] and iii) Complete unitarization of the  $\pi\pi$  intermediate states through a Khuri-Treiman treatment and inclusion of the experimental  $\gamma\gamma \rightarrow \pi^0\pi^0$  amplitude [13]. It has to be emphasized that these higher order corrections do not represent a complete chiral order contribution. The size of the first correction is controversial [7, 8], however a large contribution by itself is excluded by the experimental spectrum. The second contribution shows that unitarity corrections of  $K_L \rightarrow \pi^0\pi^+\pi^-$  to  $K_L \rightarrow \pi^0\gamma\gamma$  increase 20 – 30% the leading order amplitude and then have to be taken into account [11, 12]. When both contributions are added it was possible to fit both the width and the spectrum [12]. The third correction increases the branching ratio by an additional 10 – 20%.

The charged channel  $K^+ \rightarrow \pi^+ \gamma \gamma$  is experimentally less known (only an upper limit on the branching ratio exists [14]), but measurements with good precision are foreseen in the near future [3, 15]. The leading one-loop  $\mathcal{O}(p^4)$  result has been computed [5] and depends upon unknown weak local amplitudes which however could give important informations on vector meson weak interactions. Indeed while at  $\mathcal{O}(p^4)$  it is well known that the finite part of the counterterms in the strong  $\chi PT$  lagrangian is saturated by the spectrum of lightest resonances (vector, axial-vector, scalar and pseudoscalar) [16], the situation in the  $\mathcal{O}(p^4)$  weak lagrangian is less clear due to our ignorance on the weak couplings of vector mesons, however there are predictive models to test [17].

We have evaluated the  $K^+ \rightarrow \pi^+ \pi^+ \pi^-$  unitarity contribution to  $K^+ \rightarrow \pi^+ \gamma \gamma$  with the same procedure used to compute the  $K_L \rightarrow \pi^0 \pi^+ \pi^-$  unitarity corrections to  $K_L \rightarrow \pi^0 \gamma \gamma$  [11]. The dispersive contribution is unambiguously computed up to a polynomial piece which is reabsorbed (together with the loop divergences) in the counterterms. Moreover various vector dominance exchange models are studied to saturate the bulk of the counterterms as in  $K_L \rightarrow \pi^0 \gamma \gamma$ . Then we can give a definite prediction for the kinematical region of nearly collinear photons where the unknown contribution of the  $\mathcal{O}(p^4)$  local amplitude is negligible. The experimental study of  $K^+ \rightarrow \pi^+ \gamma \gamma$  in the DAΦNE  $\Phi$ -factory [3] and other experiments [15] is going to improve the phenomenology of  $K \rightarrow \pi \gamma \gamma$  and will allow us to analyze both channels in a correlated way.

## 2 $K \rightarrow \pi \gamma \gamma$ amplitudes

The general amplitude for  $K \rightarrow \pi \gamma \gamma$  is given by

$$M ( K(p) \rightarrow \pi(p_3) \gamma(q_1, \epsilon_1) \gamma(q_2, \epsilon_2) ) = \epsilon_{1\mu} \epsilon_{2\nu} M^{\mu\nu}(p, q_1, q_2) \quad (1)$$

where  $\epsilon_1, \epsilon_2$  are the photon polarizations, and  $M^{\mu\nu}$  has four invariant amplitudes

$$\begin{aligned} M^{\mu\nu} = & \frac{A(z, y)}{m_K^2} (q_2^\mu q_1^\nu - q_1 \cdot q_2 g^{\mu\nu}) + \frac{2 B(z, y)}{m_K^4} (-p \cdot q_1 p \cdot q_2 g^{\mu\nu} - q_1 \cdot q_2 p^\mu p^\nu \\ & + p \cdot q_1 q_2^\mu p^\nu + p \cdot q_2 p^\mu q_1^\nu) \\ & + \frac{C(z, y)}{m_K^2} \varepsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma} + \frac{D(z, y)}{m_K^4} [ \varepsilon^{\mu\nu\rho\sigma} (p \cdot q_2 q_{1\rho} + p \cdot q_1 q_{2\rho}) p_\sigma \\ & + (p^\mu \varepsilon^{\nu\alpha\beta\gamma} + p^\nu \varepsilon^{\mu\alpha\beta\gamma}) p_\alpha q_{1\beta} q_{2\gamma} ] \end{aligned} \quad (2)$$

where

$$y = \frac{p \cdot (q_1 - q_2)}{m_K^2}, \quad z = \frac{(q_1 + q_2)^2}{m_K^2}. \quad (3)$$

The physical region in the adimensional variables  $y$  and  $z$  is given by :

$$0 \leq |y| \leq \frac{1}{2} \lambda^{1/2} (1, r_\pi^2, z) \quad , \quad 0 \leq z \leq (1 - r_\pi)^2 \quad , \quad (4)$$

with

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) \quad , \quad (5)$$

$$r_\pi = \frac{m_\pi}{m_K} \quad .$$

Note that the invariant amplitudes  $A(z, y)$ ,  $B(z, y)$  and  $C(z, y)$  have to be symmetric under the interchange of  $q_1$  and  $q_2$  as required by Bose symmetry, while  $D(z, y)$  is anti-symmetric. In the limit where CP is conserved the amplitudes  $A$  and  $B$  contribute only to  $K_L \rightarrow \pi^0 \gamma \gamma$ , while  $C$  and  $D$  only contribute to  $K_S \rightarrow \pi^0 \gamma \gamma$ . In  $K^+ \rightarrow \pi^+ \gamma \gamma$  all of them are involved.

Using the definitions (2,3) the double differential rate for unpolarized photons is given by

$$\begin{aligned} \frac{\partial^2 \Gamma}{\partial y \partial z} = & \frac{m_K}{2^9 \pi^3} \left[ z^2 (|A + B|^2 + |C|^2) \right. \\ & \left. + \left( y^2 - \frac{1}{4} \lambda(1, r_\pi^2, z) \right)^2 (|B|^2 + |D|^2) \right] \quad . \end{aligned} \quad (6)$$

The processes  $K \rightarrow \pi \gamma \gamma$  have no tree level  $\mathcal{O}(p^2)$  contribution. At  $\mathcal{O}(p^4)$  the amplitudes  $B(z, y)$  and  $D(z, y)$  are still zero since there are not enough powers of momenta to generate the gauge structure, and therefore their leading contribution is  $\mathcal{O}(p^6)$ . As can be seen from (6) only the  $B$  and  $D$  terms contribute for small  $z$  (the invariant amplitudes are regular in the small  $y, z$  region). The antisymmetric character of the  $D(z, y)$  amplitude under the interchange of  $q_1$  and  $q_2$  means effectively that while its leading contribution is  $\mathcal{O}(p^6)$  this only can come from a finite loop calculation because the leading counterterms for the  $D$  amplitude are  $\mathcal{O}(p^8)$ . However also this loop contribution is helicity suppressed compared to the  $B$  term. As shown in a similar situation in the electric Direct Emission of  $K_L \rightarrow \pi^+ \pi^- \gamma$  [18], this antisymmetric  $\mathcal{O}(p^6)$  loop contribution might be smaller than the local  $\mathcal{O}(p^8)$  contribution.

### 3 $K^+ \rightarrow \pi^+ \gamma \gamma$ at $\mathcal{O}(p^4)$

The leading  $\Delta I = 1/2$   $\mathcal{O}(p^4)$   $A(z, y)$  and  $C(z, y)$  amplitudes for  $K^+ \rightarrow \pi^+ \gamma \gamma$  have already been computed in [5]. We review them here.

The  $A$  amplitude reads

$$A^{(4)}(z) = \frac{G_8 m_K^2 \alpha_{em}}{2\pi z} \left[ (z + 1 - r_\pi^2) F\left(\frac{z}{r_\pi^2}\right) + (z + r_\pi^2 - 1) F(z) - \hat{c}z \right] \quad . \quad (7)$$

Here  $G_8$  is the effective weak coupling constant determined from  $K \rightarrow \pi \pi$  decays at  $\mathcal{O}(p^2)$

$$|G_8| \simeq 9.2 \times 10^{-6} \text{ GeV}^{-2} \quad (8)$$

and the  $F(x)$  function is defined as

$$F(x) = \begin{cases} 1 - \frac{4}{x} \arcsin^2\left(\frac{\sqrt{x}}{2}\right) & , x \leq 4 \\ 1 + \frac{1}{x} \left( \ln^2\left(\frac{1-\beta(x)}{1+\beta(x)}\right) - \pi^2 + 2i\pi \ln\left(\frac{1-\beta(x)}{1+\beta(x)}\right) \right) & , x > 4 \end{cases} \quad (9)$$

$$\beta(x) = \sqrt{1 - \frac{4}{x}} \quad .$$

In (7) the pion loop contribution  $F(z/r_\pi^2)$  dominates by far over the kaon loop amplitude with  $F(z)$ . The loop results are finite. However as we have already commented  $\chi PT$  allows an  $\mathcal{O}(p^4)$  scale independent local contribution that in (7) is parameterized by

$$\hat{c} = \frac{128\pi^2}{3} [3(L_9 + L_{10}) + N_{14} - N_{15} - 2N_{18}] \quad , \quad (10)$$

that is a quantity  $\mathcal{O}(1)$ . The  $L_9$  and  $L_{10}$  are the local  $\mathcal{O}(p^4)$  strong couplings and  $N_{14}, N_{15}$  and  $N_{18}$  are  $\mathcal{O}(p^4)$  weak couplings, still not fixed by the phenomenology, and that can be only computed in a model dependent way [17]. The Weak Deformation Model (WDM) [8] predicts  $\hat{c} = 0$ , while naive factorization in the Factorization Model (FM) [19, 17] gives  $\hat{c} = -2.3$ . In these models, due to the cancellation in the vector meson contribution in (10), the role of axial mesons could be relevant [17].

The  $\mathcal{O}(p^4)$  contribution to the  $C(z, y)$  amplitude is

$$C(z) = \frac{G_8 m_K^2 \alpha_{em}}{\pi} \left[ \frac{z - r_\pi^2}{z - r_\pi^2 + i r_\pi \frac{\Gamma_{\pi^0}}{m_K}} - \frac{z - \frac{2 + r_\pi^2}{3}}{z - r_\eta^2} \right] \quad , \quad (11)$$

where  $r_\eta = m_\eta/m_K$  and  $\Gamma_{\pi^0} \equiv \Gamma(\pi^0 \rightarrow \gamma\gamma)$ . This amplitude is generated by the Wess–Zumino–Witten functional [20]  $(\pi^0, \eta) \rightarrow \gamma\gamma$  through the sequence  $K^+ \rightarrow \pi^+(\pi^0, \eta) \rightarrow \pi^+\gamma\gamma$ . This contribution amounts roughly to less than 10% in the total width.

## 4 $\mathcal{O}(p^6)$ local amplitudes for $K^+ \rightarrow \pi^+\gamma\gamma$

At  $\mathcal{O}(p^6)$  there are only four independent Lorentz invariant local amplitudes contributing to  $K^+ \rightarrow \pi^+\gamma\gamma$  :

$$F_{\mu\nu} F^{\mu\lambda} \partial^\nu K^+ \partial_\lambda \pi^- \quad , \quad F_{\mu\nu} F^{\mu\nu} \partial_\lambda K^+ \partial^\lambda \pi^- \quad , \quad (12)$$

$$m_K^2 F_{\mu\nu} F^{\mu\nu} K^+ \pi^- \quad , \quad \partial^\alpha F_{\mu\nu} \partial_\alpha F^{\mu\nu} K^+ \pi^- \quad ,$$

where the last one has no analogous in  $K_L \rightarrow \pi^0\gamma\gamma$ . In general the couplings of these operators are not directly related with the ones in  $K_L \rightarrow \pi^0\gamma\gamma$  due to the fact that the

electric charge matrix does not commute with the generators of the electrically charged field. As we shall see, in specific models, relations between the two channels can be found. All of these operators contribute to  $A(z, y)$  but only the first one in (12) gives a  $B(z, y)$  amplitude. Loop divergences at  $\mathcal{O}(p^6)$  are absorbed in the counterterm coefficients, that thus renormalized are finite. Chiral dimensional analysis [1] tells us that their contributions are suppressed compared to  $\mathcal{O}(p^4)$  by a factor  $m_K^2/(4\pi F_\pi)^2 \sim 0.2$ . Nevertheless vector meson exchange was found to enhance this up to  $m_K^2/m_V^2 \sim 0.4$  [16]. Thus we try to estimate the contributions of the lightest resonances, i.e. vector mesons, and we assume that heavier resonances and non-resonant contributions give smaller corrections. This picture seems well verified in the strong coupling constants at  $\mathcal{O}(p^4)$  [16], and it is likely to apply to the weak couplings.

We are going to consider here the contribution of vector meson resonances to the weak  $\mathcal{O}(p^6)$  counterterms for  $K^+ \rightarrow \pi^+ \gamma \gamma$  (a more complete discussion and an extension to  $K_L \rightarrow \pi^0 \gamma \gamma$  is given in [21]) and we find that only the terms in (12) with derivatives on the meson fields can be generated by vector meson exchange.

Following the study of  $K_L \rightarrow \pi^0 \gamma \gamma$  in [8] we assume that the contribution to the local amplitudes for  $K^+ \rightarrow \pi^+ \gamma \gamma$  is dominated by vector meson resonances and define their contribution with an adimensional parameter  $a_V$  generated by the first term in (12) as

$$a_V = - \frac{\pi}{2G_8 m_K^2 \alpha_{em}} \lim_{z \rightarrow 0} B_V(z) \quad (13)$$

where  $B_V(z)$  is the vector resonance contribution to the  $B$  amplitude. The  $A$  amplitude generated by vector exchange ( $A_V$ ) gets contributions from the first and second structure in (12). However as seen in [8] if we assume that these local amplitudes are generated through strong resonance exchange supplemented with a weak transition in the external legs, those two contributions are related and can be written in terms of the  $a_V$  parameter defined in (13) as

$$A_V = \frac{G_8 m_K^2 \alpha_{em}}{\pi} a_V (3 + r_\pi^2 - z) \quad (14)$$

In [8] two different vector contributions to  $a_V$  in (13) have been proposed : the first one amounts for the weak counterterms generated by a strong vector resonance exchange with a weak transition in an external leg ( $a_V^{ext}$ ), the second is generated by vector resonance exchange between a direct weak vector-pseudoscalar-photon ( $VP\gamma$ ) vertex and a strong one ( $a_V^{dir}$ ). While the first can be computed in a model independent way, the generation of direct  $VP\gamma$  weak vertices is still poorly known and therefore only models can be used.

The *external* weak transition for  $K^+ \rightarrow \pi^+ \gamma \gamma$  gives

$$a_V^{ext} = - \frac{128\pi^2 h_V^2 m_K^2}{9 m_V^2} = -0.08 \quad (15)$$

where  $m_V = m_\rho$  and the strong  $VP\gamma$  coupling  $|h_V| = (3.7 \pm 0.3) \times 10^{-2}$  is defined by

$$\mathcal{L}(VP\gamma) = h_V \varepsilon_{\mu\nu\rho\sigma} \langle V^\mu \{ u^\nu, f_+^{\rho\sigma} \} \rangle \quad , \quad (16)$$

with

$$u_\mu = iu^\dagger D_\mu U u^\dagger \quad , \quad f_+^{\mu\nu} = u F_L^{\mu\nu} u^\dagger + u^\dagger F_R^{\mu\nu} u \quad ,$$

$$U = uu \quad , \quad D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu \quad , \quad (17)$$

$$U = \exp \left( \frac{i}{F_\pi} \sum_{j=1}^8 \lambda_j \phi_j \right) \quad ,$$

and  $u(\phi)$  is an element of the coset space  $SU(3)_L \otimes SU(3)_R / SU(3)_V$  parameterized in terms of the Goldstone fields  $\phi_i$   $i = 1, \dots, 8$ ,  $F_{R,L}^{\mu\nu}$  are the strength field tensors associated to the external  $r_\mu$  and  $\ell_\mu$  fields,  $V_\mu$  is the nonet of vector meson resonances,  $F_\pi \simeq 93 \text{ MeV}$  is the decay constant of pion and, finally,  $\langle A \rangle \equiv \text{Tr}(A)$ . In our case with two photon external fields  $\ell_\mu = r_\mu = eQ A_\mu$ <sup>1</sup>.

The model dependent contribution to  $a_V$  from *direct* weak vertices in the WDM and FM quoted above is :

$$a_{V,WDM}^{dir} = -a_V^{ext} \quad , \quad (18)$$

$$a_{V,FM}^{dir} = -2k_F a_V^{ext} \quad ,$$

where in the FM  $k_F$  is the unknown fudge factor that is not fixed by the model and satisfies  $0 < k_F \leq 1$ . Naive factorization predicts  $k_F = 1$ .

By adding (15) and (18) we see that  $a_V = a_V^{ext} + a_V^{dir}$  is

$$a_{V,WDM} = 0 \quad , \quad (19)$$

$$a_{V,FM} = (1 - 2k_F) a_V^{ext} \quad .$$

We note that for the allowed range of values of  $k_F$  in the FM both models agree in predicting a very small vector meson contribution to  $\mathcal{O}(p^6)$  in  $K^+ \rightarrow \pi^+ \gamma \gamma$  and therefore the important conclusion of our exercise is that the vector contribution to the local  $B(z, y)$  amplitude can be neglected. If, as it is reasonable to assume, other resonance interchange corrections are even smaller, we can conclude that the small  $z$  region of  $K^+ \rightarrow \pi^+ \gamma \gamma$  is completely predictable at  $\mathcal{O}(p^6)$  through chiral loops and it is likely to be dominated by the unitarity corrections of  $K^+ \rightarrow \pi^+ \pi^+ \pi^-$  to  $K^+ \rightarrow \pi^+ \gamma \gamma$ . Indeed in that region the  $\mathcal{O}(p^6)$  is dominant and our ignorance on the  $\hat{c}$  amplitude is irrelevant.

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<sup>1</sup>For a full discussion of the definitions and notations see [16, 8].

## 5 Unitarity corrections of $K^+ \rightarrow \pi^+\pi^+\pi^-$ to $K^+ \rightarrow \pi^+\gamma\gamma$

The amplitude for the process  $K(p) \rightarrow \pi(p_1)\pi(p_2)\pi(p_3)$  can be expanded in powers of the Dalitz plot variables

$$X = \frac{s_2 - s_1}{m_\pi^2} \quad , \quad Y = \frac{s_3 - s_o}{m_\pi^2} \quad , \quad (20)$$

where  $s_i = (p - p_i)^2$  for  $i = 1, 2, 3$ ,  $s_o = (s_1 + s_2 + s_3)/3$  and the subscript 3 indicates the odd pion. For the decay  $K^+(p) \rightarrow \pi^+(p_1)\pi^+(p_2)\pi^-(p_3)$  the isospin decomposition, neglecting the small phase shifts and up to quadratic terms, is written as [22, 23]

$$\begin{aligned} A(K^+ \rightarrow \pi^+\pi^+\pi^-) = & 2\alpha_1 - \alpha_3 + (\beta_1 - \frac{1}{2}\beta_3 + \sqrt{3}\gamma_3)Y \\ & - 2(\zeta_1 + \zeta_3)(Y^2 + \frac{X^2}{3}) - (\xi_1 + \xi_3 - \xi'_3)(Y^2 - \frac{X^2}{3}) \quad , \end{aligned} \quad (21)$$

where the subscripts 1 and 3 refer to  $\Delta I = 1/2, 3/2$  transitions respectively, and the coefficients in (21) have been fitted to the data [23]. The  $\mathcal{O}(p^2)$  amplitude

$$A^{(2)}(K^+ \rightarrow \pi^+\pi^+\pi^-) = G_8 m_K^2 \left( \frac{2}{3} - r_\pi^2 Y \right) \quad (22)$$

with the value of  $G_8$  in (8) underestimates by 20-30% the experimental linear slopes for  $K^+ \rightarrow \pi^+\pi^+\pi^-$ . At the next chiral order, the experimental linear and quadratic slopes in (21) are recovered (with predictive power too) [23]. Since the  $\mathcal{O}(p^4)$  loop contribution (7) to  $K^+ \rightarrow \pi^+\gamma\gamma$ , is generated by (22), it seems natural to try to include all the contributions to  $K^+ \rightarrow \pi^+\gamma\gamma$  generated by the experimental slopes in (21), i.e. we evaluate the  $\mathcal{O}(p^6)$  contributions to  $K^+ \rightarrow \pi^+\gamma\gamma$  induced by the  $\mathcal{O}(p^4)$  corrections to  $K^+ \rightarrow \pi^+\pi^+\pi^-$ . This can be done similarly to the case of  $K_L \rightarrow \pi^0\gamma\gamma$  in [11, 12]: (21) is considered as an effective  $K^+ \rightarrow \pi^+\pi^+\pi^-$  chiral vertex, the kinematical variables are replaced by the appropriate covariant derivatives, the QED scalar vertices are added through minimal coupling and then the usual Feynman diagrams approach can be used. There are 13 Feynman diagrams that have two different topologies. There is a subset of 9 diagrams where one or both photons are radiated by the external legs and therefore do not give an absorptive contribution. The sum of these bremsstrahlung-like diagrams is not gauge invariant and cancels with an analogous term from the 4 remaining diagrams (two of which have also an absorptive part) and generate the  $A(z, y)$  and  $B(z, y)$  amplitudes. This loop result for the  $A(z, y)$  and  $B(z, y)$  amplitudes is divergent and needs to be regularized and renormalized. We have used dimensional regularization and divergences are absorbed by the four counterterms discussed earlier.

Alternatively one could use a subtracted dispersion relation. While the absorptive contribution is uniquely determined, the dispersive one, due to the presence of subtraction



constants, can be computed only up to a polynomial. This is precisely the same ambiguity that happens in the effective vertex method that we have used where the finite part of the counterterms is a priori unknown.

We assume that the dominant contribution should come from the non-polynomial amplitude generated by the cut of the two-pion intermediate state. Otherwise contributions generated by a vanishing on-shell  $K \rightarrow 3\pi$  amplitude can be reabsorbed in the unknown polynomial amplitude.

The final result for the  $A$  and  $B$  amplitudes for  $K^+ \rightarrow \pi^+ \gamma \gamma$  in the  $\overline{MS}$  subtraction scheme are

$$\begin{aligned}
A(z, y) = & \frac{\alpha_{em}}{2\pi} \cdot \\
& \left\{ \left[ 2(2\alpha_1 - \alpha_3) + \left( 1 + \frac{1}{3r_\pi^2} - \frac{z}{r_\pi^2} \right) \left( \beta_1 - \frac{1}{2}\beta_3 + \sqrt{3}\gamma_3 \right) \right] \frac{1}{z} F\left(\frac{z}{r_\pi^2}\right) \right. \\
& - \frac{8}{3r_\pi^4} (2\zeta_1 - \xi_1) \left[ r_\pi^2 \left( \ln\left(\frac{m_\pi^2}{\mu^2}\right) - 1 \right) \right. \\
& \quad \left. \left. + \frac{1}{18} (1 + 6(r_\pi^2 - z) + 9(r_\pi^2 - z)^2) \frac{1}{z} F\left(\frac{z}{r_\pi^2}\right) \right] \right. \\
& - \frac{8}{3r_\pi^4} (4\zeta_1 + \xi_1) \left[ -\frac{1}{12} (1 + 6r_\pi^2) \ln\left(\frac{m_\pi^2}{\mu^2}\right) + \frac{r_\pi^2}{2} \right. \\
& \quad - \frac{1}{36} (9r_\pi^2 - 5 - 3(1 + 3r_\pi^2)(r_\pi^2 - z)) \frac{1}{z} F\left(\frac{z}{r_\pi^2}\right) \\
& \quad + \frac{y^2}{z} \left( \frac{1}{12} + 3R\left(\frac{z}{r_\pi^2}\right) + \frac{1}{2} \left( 1 + \frac{2r_\pi^2}{z} \right) F\left(\frac{z}{r_\pi^2}\right) \right) \\
& \quad - \frac{(1 - r_\pi^2 + z)^2}{4z} \left( \frac{1}{12} + R\left(\frac{z}{r_\pi^2}\right) + \frac{1}{2} \left( 1 + \frac{2r_\pi^2}{z} \right) F\left(\frac{z}{r_\pi^2}\right) \right) \\
& \quad + (1 - r_\pi^2 + z) \left( \frac{1}{24} + \frac{z}{72r_\pi^2} \right. \\
& \quad \quad \left. + \frac{1}{12} \left( 1 + \frac{2r_\pi^2}{z} \right) \left( \frac{z}{r_\pi^2} R\left(\frac{z}{r_\pi^2}\right) + 3F\left(\frac{z}{r_\pi^2}\right) \right) \right) \\
& \quad - z \left( \frac{1}{36} - \frac{r_\pi^2}{24z} + \frac{z}{72r_\pi^2} + \frac{1}{12} \left( \frac{z}{r_\pi^2} + 1 - \frac{2r_\pi^2}{z} \right) R\left(\frac{z}{r_\pi^2}\right) \right. \\
& \quad \left. \left. + \frac{r_\pi^2}{2z} \left( 1 - \frac{r_\pi^2}{z} \right) F\left(\frac{z}{r_\pi^2}\right) \right) \right] \\
& + G_8 m_K^2 \left[ (z + r_\pi^2 - 1) \frac{1}{z} F(z) - \hat{c} + 2r_\pi^2 \eta_1 + 2\eta_2 \right] \Big\} , \tag{23}
\end{aligned}$$

$$\begin{aligned}
B(z, y) = & \frac{\alpha_{em}}{\pi} \left\{ \frac{1}{3r_\pi^4} (4\zeta_1 + \xi_1) \left[ -\frac{1}{6} \left( 1 + 2 \ln \left( \frac{m_\pi^2}{\mu^2} \right) \right) + \frac{z}{18r_\pi^2} - \frac{2r_\pi^2}{z} F \left( \frac{z}{r_\pi^2} \right) \right. \right. \\
& \left. \left. + \frac{1}{3} \left( \frac{z}{r_\pi^2} - 10 \right) R \left( \frac{z}{r_\pi^2} \right) \right] \right. \\
& \left. + G_8 m_K^2 \eta_3 \right\} \quad , \quad (24)
\end{aligned}$$

where the  $F(x)$  function has been defined in (9) and

$$R(x) = \begin{cases} -\frac{1}{6} + \frac{2}{x} - \frac{2}{x} \sqrt{\frac{4}{x} - 1} \arcsin\left(\frac{\sqrt{x}}{2}\right) & , \quad x \leq 4 \\ -\frac{1}{6} + \frac{2}{x} + \frac{\beta(x)}{x} \left( \ln \left( \frac{1 - \beta(x)}{1 + \beta(x)} \right) + i\pi \right) & , \quad x > 4 \end{cases} \quad (25)$$

with  $\beta(x)$  defined in (9) ( $F(z) \simeq -\frac{z}{12}$  and  $R(z) \simeq \frac{z}{60}$  for  $z \rightarrow 0$ ).

In (23,24) the  $\eta_i$ ,  $i = 1, 2, 3$  stand for the unknown polynomial contribution we have spoken before. Though, in principle, four different subtraction constants could appear, these are the only ones that are necessary in order to absorb the loop divergences generated by this unitarity correction. We emphasize that these local amplitudes are not generated by the vector mesons (already taken into account by  $a_V$ ) and therefore are expected to be suppressed by  $m_K^2/\Lambda_\chi^2$  with  $\Lambda_\chi \simeq 4\pi F_\pi$  over the previous chiral order. If as a naive chiral dimensional analysis we choose as coefficients of the suppression factor the factor accompanying  $\ln(m_\pi^2/\mu^2)$  (for definiteness) we get

$$\eta_1 = 12 \frac{m_K^2}{\Lambda_\chi^2} \simeq 2.06 \quad , \quad \eta_2 = \frac{7}{5} \frac{m_K^2}{\Lambda_\chi^2} \simeq 0.24 \quad , \quad \eta_3 = -\frac{3}{2} \frac{m_K^2}{\Lambda_\chi^2} \simeq -0.26 \quad . \quad (26)$$

We note the big numerical value for  $\eta_1$ , but looking into the  $A$  amplitude (23) we see that  $\eta_1$  is suppressed by  $m_\pi^2/m_K^2$ . The order of magnitude we get indeed is the one expected by chiral counting. In (23,24) we only have included the  $\Delta I = 3/2$  coefficients of the  $\mathcal{O}(p^2)$  contribution of  $K^+ \rightarrow \pi^+ \pi^+ \pi^-$  because the bigger errors in the determination of these coefficients in the  $\mathcal{O}(p^4)$  amplitudes.

We notice that at this order while the  $B$  amplitude is  $y$ -independent the  $A$  amplitude gets its first  $y$  dependence.

As we have seen the vector meson contribution to the  $B$  amplitude is likely to be very small. If we use the same definition (13) for our  $B$  amplitude (24) we get

$$a_\ell = \frac{4\zeta_1 + \xi_1}{18G_8 m_K^2 r_\pi^4} \ln \left( \frac{m_\pi^2}{\mu^2} \right) - \frac{1}{2} \eta_3 \quad . \quad (27)$$

The first term amounts to  $\simeq 0.52$  if  $\mu = m_\rho$  is taken. Then from (26) we can conclude that due to the expected vanishing of the vector meson contributions, the polynomial non-resonant amplitude, though small compared with the unitarity corrections, could be

in principle tested in this channel at low  $z$ . We note that this situation is different to the  $K_L \rightarrow \pi^0 \gamma \gamma$  case where the vector meson contribution seems to be more relevant.

We have not included the  $\Delta I = 3/2$  couplings in the  $\mathcal{O}(p^6)$  contribution because of the big errors in the experimental fit. But we should stress that inputting those amplitudes could modify the low- $z$  behaviour of the  $B$  amplitude even a 30%.

The argument about the relevance of the correction to  $K_L \rightarrow \pi^0 \gamma \gamma$  coming from the inclusion of the experimental  $\gamma \gamma \rightarrow \pi^0 \pi^0$  amplitude [13] is weakened here due to the relative suppression of  $K^+ \rightarrow \pi^+ \pi^0 \pi^0$  compared to  $K^+ \rightarrow \pi^+ \pi^+ \pi^-$ .

## 6 $K^+ \rightarrow \pi^+ \gamma \gamma$ : branching ratio and spectra

We can now proceed to study the numerical results for  $K^+ \rightarrow \pi^+ \gamma \gamma$  taking into account the  $\mathcal{O}(p^4)$  and our  $\mathcal{O}(p^6)$  loop evaluation of the unitarity corrections. As we have shown the  $\mathcal{O}(p^6)$  vector resonance dominated local contributions are negligible. In our numerical discussion we will input  $\eta_i = 0$ ,  $i = 1, 2, 3$  and  $\mu = m_\rho$ .

At present there is only an upper bound for  $\text{Br}(K^+ \rightarrow \pi^+ \gamma \gamma)$  which depends on the shape of the spectrum [24]

$$\begin{aligned} \text{Br}(K^+ \rightarrow \pi^+ \gamma \gamma) |_{exp} &\leq 1.5 \times 10^{-4} \quad (\chi PT \text{ amplitude}) \quad , \\ \text{Br}(K^+ \rightarrow \pi^+ \gamma \gamma) |_{exp} &\leq 1.0 \times 10^{-6} \quad (\text{constant amplitude}) \quad . \end{aligned} \tag{28}$$

The uncertainty in the theoretical prediction is dominated by the unknown  $\mathcal{O}(p^4)$  counterterm generated amplitude  $\hat{c}$  in (7). In Fig. 1 we show  $\text{Br}(K^+ \rightarrow \pi^+ \gamma \gamma)$  as a function of  $\hat{c}$  with and without the  $\mathcal{O}(p^6)$  corrections that we have computed. We remind that WDM predicts  $\hat{c} = 0$  while naive FM gives  $\hat{c} = -2.3$  with the following results :

$$\begin{aligned} \text{Br}(K^+ \rightarrow \pi^+ \gamma \gamma) |_{WDM} &= 7.24 \times 10^{-7} \quad , \\ \text{Br}(K^+ \rightarrow \pi^+ \gamma \gamma) |_{nFM} &= 6.20 \times 10^{-7} \quad . \end{aligned} \tag{29}$$

When comparing with the  $\mathcal{O}(p^4)$  predictions [5] we find that the unitarity corrections increase around 30 – 40% the branching ratio. In Fig. 2 we show the  $z$ -distribution at  $\mathcal{O}(p^4)$  and with our  $\mathcal{O}(p^6)$  unitarity correction for  $\hat{c} = 0$ . As can be seen the correction is noticeable.

The uncertainty on the  $\hat{c}$  amplitude also translates into the spectra. In Fig. 3 we show the spectrum of the invariant mass of the two photons for the two values of  $\hat{c}$  predicted by the WDM and naive FM. We notice that the main dependence on  $\hat{c}$  arises in the  $z$ -region where there is the bulk of the absorptive contribution.

Finally in Fig. 4 we show the spectrum in the “asymmetric”  $y$  variable for  $\hat{c} = 0$ . At this order this spectrum has much less structure than the one in the di-photon invariant mass.

## 7 Conclusion

The situation of the chiral prediction for  $K_L \rightarrow \pi^0 \gamma \gamma$  and the expected experimental measurement of  $\Gamma(K^+ \rightarrow \pi^+ \gamma \gamma)$  at DAΦNE and BNL make interesting to enlarge our theoretical knowledge about this last process.

We have reviewed and studied the vector resonance dominated  $\mathcal{O}(p^6)$  local amplitudes which main role is to determine the low- $z$  region of the invariant di-photon mass spectrum. The conclusion is that this contribution is likely to be negligible. The  $\mathcal{O}(p^6)$  unitarity corrections from  $K^+ \rightarrow \pi^+ \pi^+ \pi^-$  to  $K^+ \rightarrow \pi^+ \gamma \gamma$  have been computed and found to be relevant increasing the branching ratio (for a fixed value of the  $\hat{c}$  amplitude) around 30 – 40%. The shape of the  $z$ -distribution is shown to be sensitive to the evaluated corrections (Fig. 2 and Fig. 3).

In Fig. 1 we have shown the branching ratio for  $K^+ \rightarrow \pi^+ \gamma \gamma$  as a function of  $\hat{c}$ . The included corrections will allow to get a more accurate determination of  $\hat{c}$  once the branching ratio of  $K^+ \rightarrow \pi^+ \gamma \gamma$  is measured and therefore will provide a new independent relation between  $\mathcal{O}(p^4)$  weak coupling constants that will improve our predictive power in this sector.

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## Figure captions

**Fig. 1 :**  $Br(K^+ \rightarrow \pi^+\gamma\gamma)$  as a function of the  $\hat{c}$  amplitude. The dashed line corresponds to  $\mathcal{O}(p^4)$   $\chi$ PT amplitude. The full line corresponds to the amplitude including the  $\mathcal{O}(p^6)$  unitarity corrections.

**Fig. 2 :** Comparison of the normalized  $z$ -distribution for  $K^+ \rightarrow \pi^+\gamma\gamma$  at  $\mathcal{O}(p^4)$  (dashed line) with our  $\mathcal{O}(p^6)$  correction (full line) for  $\hat{c} = 0$ .

**Fig. 3 :** Normalized di-photon mass spectrum of  $K^+ \rightarrow \pi^+\gamma\gamma$  for  $\hat{c} = -2.3$  (naive FM, dashed line) and  $\hat{c} = 0$  (WDM, full line).

**Fig. 4 :**  $\frac{\partial Br(K^+ \rightarrow \pi^+\gamma\gamma)}{\partial y}$  spectrum for  $\hat{c} = 0$  as a function of  $|y|$ .

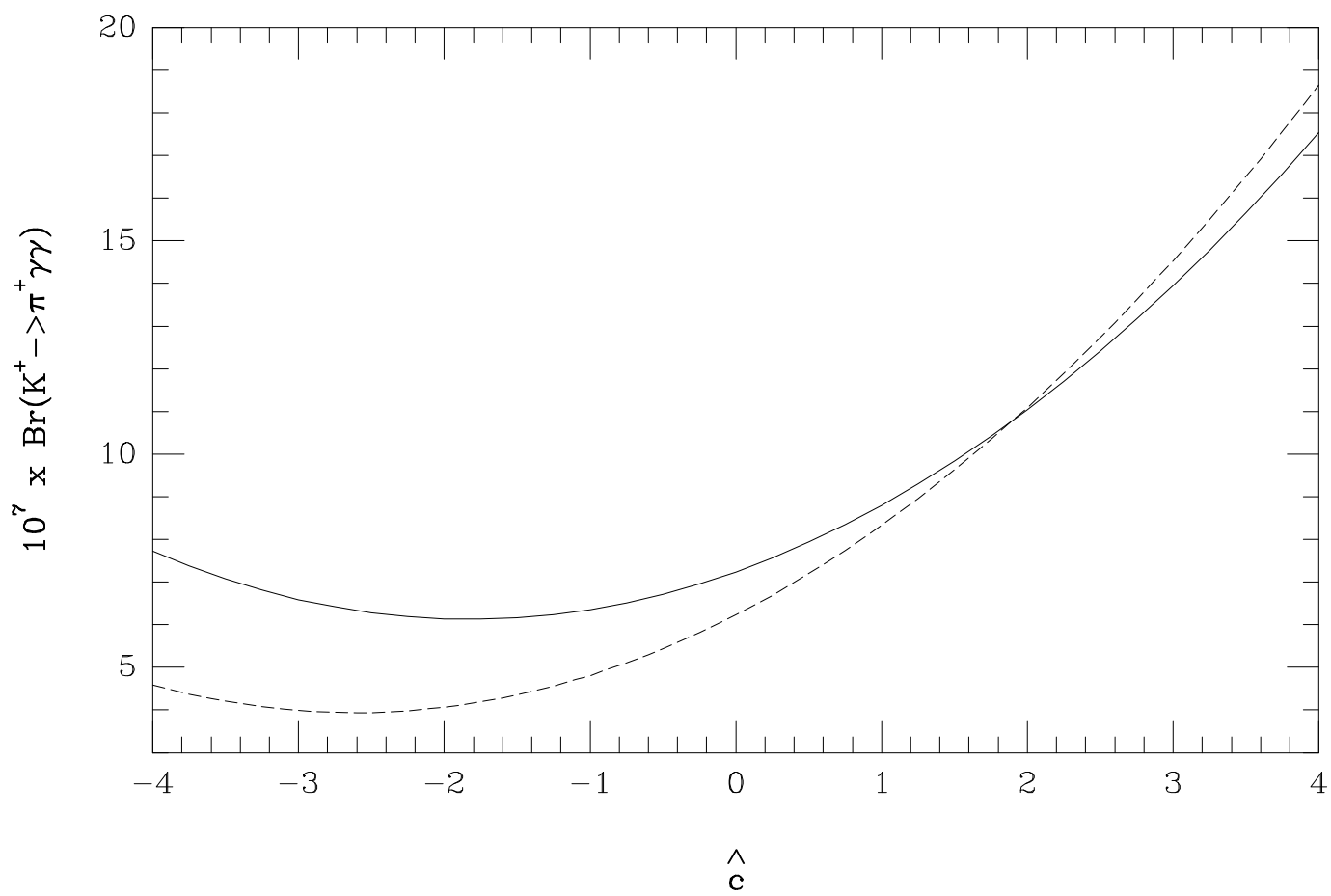


Fig. 1



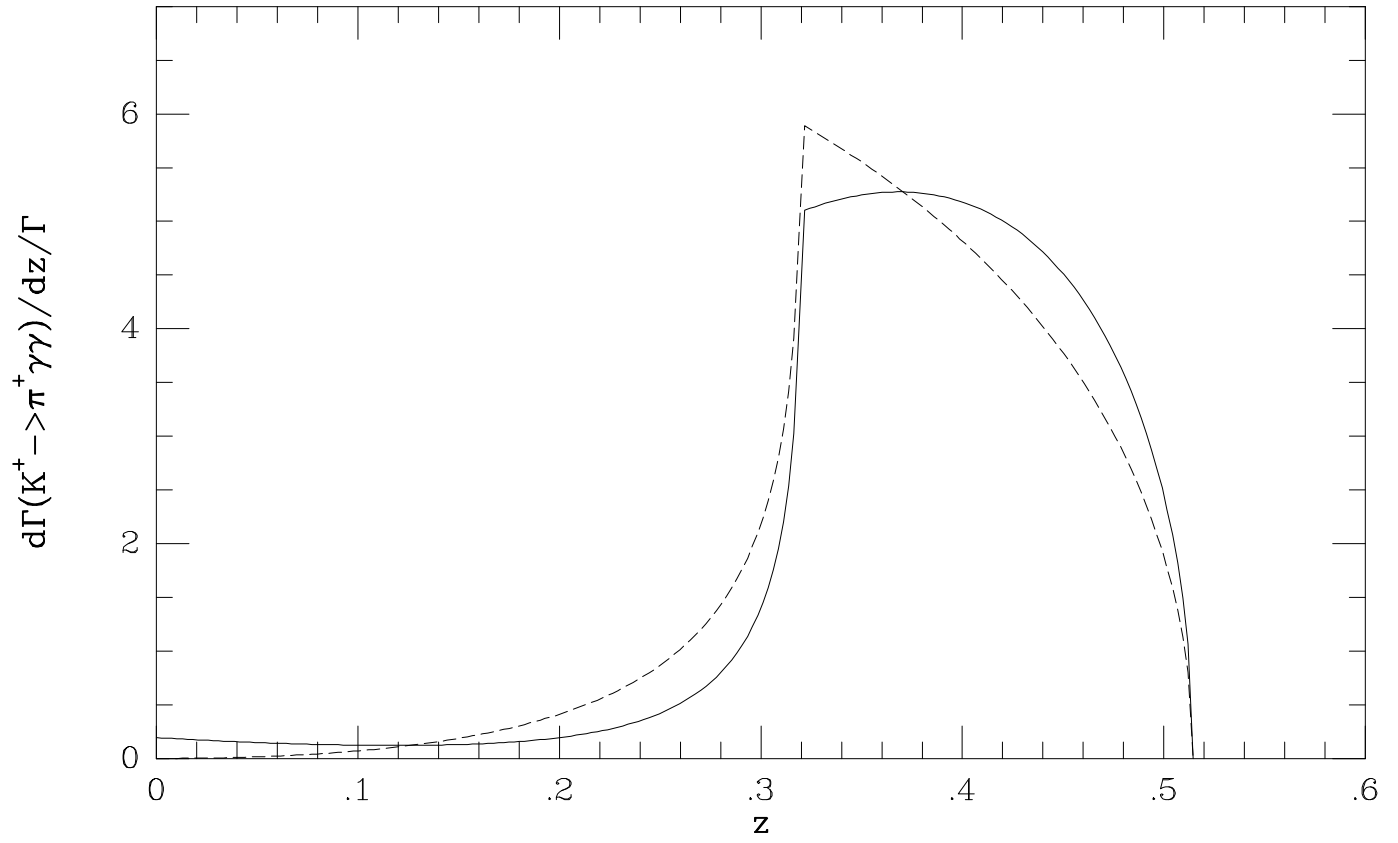


Fig. 2

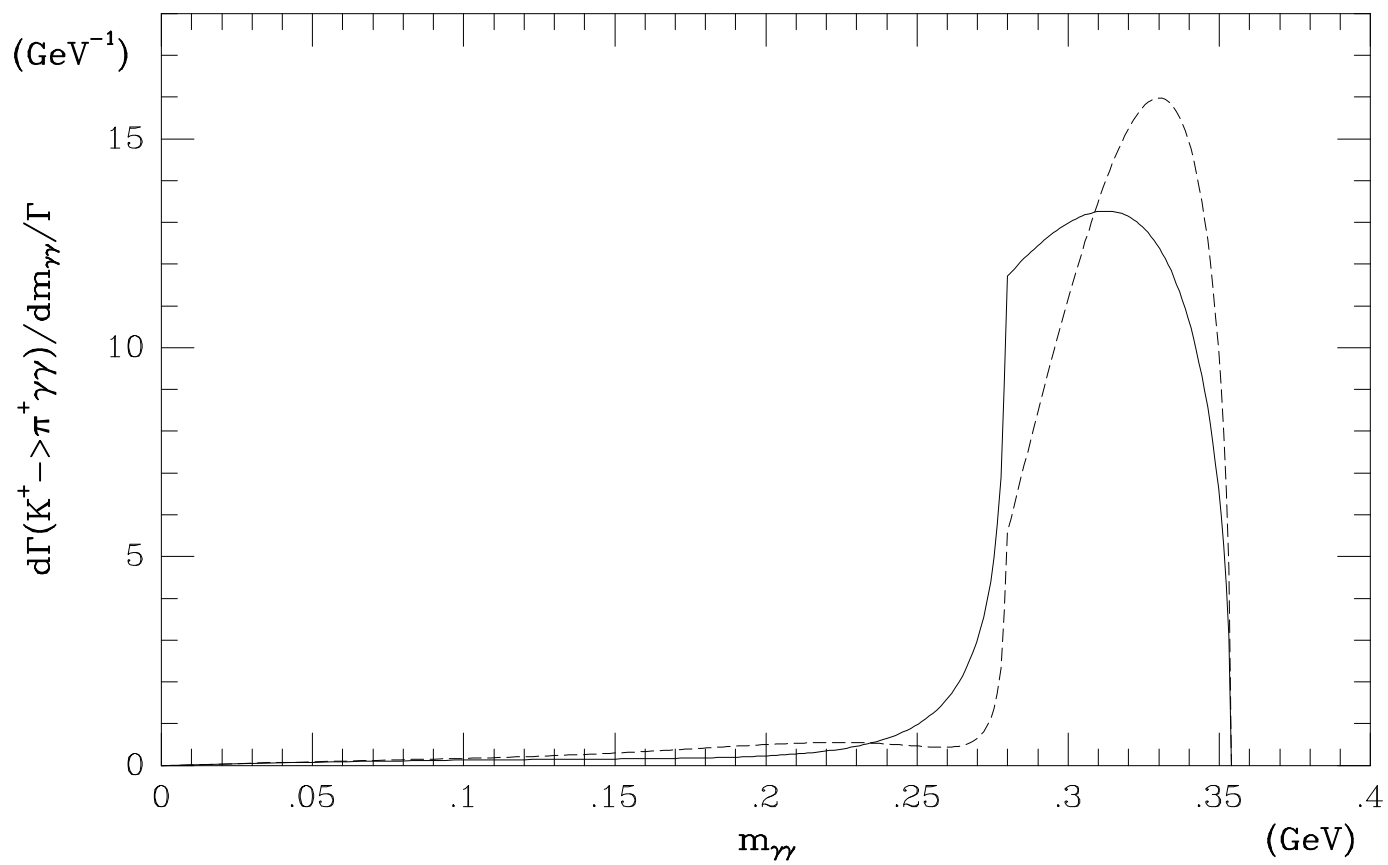


Fig. 3

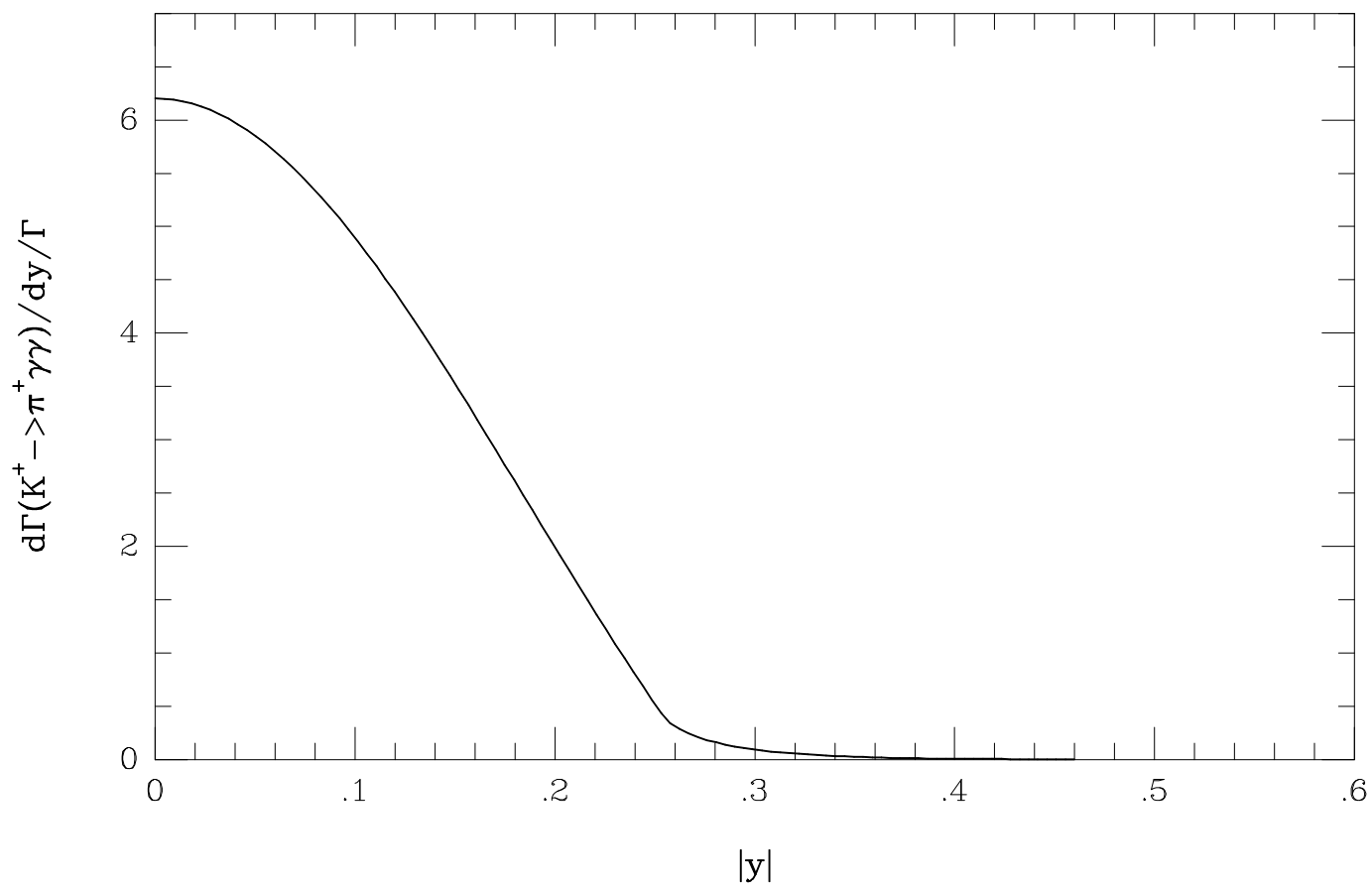


Fig. 4